

Mathematical Analysis

third level

second course

Mathematics department

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Lecher 1:

Continuity

Def: Let (X, d_1) and (Y, d_2) are two metric space.
And $S \subseteq X$, $f: S \rightarrow Y$, we say that $f(x)$ tends to S
the **Limit L** as x tend S to a

Example: let $f: \mathbb{R} \rightarrow \mathbb{R}$, s.t $f(x) = 3x - 1$, prove
that $\lim_{x \rightarrow 2} f(x) = 5$

Proof:

$$L = 5, a = 2 \quad d_1(x, y) = d_2(x, y) = |x - y|.$$

$$\forall \epsilon > 0, \text{ to find } \delta > 0, \text{ s.t } d_2(f(x), 5) < \epsilon,$$

wherever $d_1(x, 2) < \delta$

$$d_1(x, 2) = |x - 2| < \delta, \quad d_2(f(x), 5) =$$

$$|f(x) - 5| = |3x - 1 - 5|$$

$$= |3x - 6| = 3|x - 2| < 3\delta.$$

$$\text{Choose } \delta = \frac{\epsilon}{3} \rightarrow |f(x) - 5| < 3\delta = 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

$$\therefore d_2(f(x), 5) < \epsilon, \rightarrow \lim_{x \rightarrow 2} f(x) = 5$$

كيف نبرهن عن وجود نهاية:

• عندما يعطى $\lim_{x \rightarrow a} f(x) = L$

• الخطوة الأساسية التي يجب الوصول اليها هي

$$d_2(f(x), L) < \epsilon$$

• نأتي بالقيمتين a, L من السؤال المعطى

• نكون تطبيقين مسافة نسمي الأول $d_1(x, y)$ ونسمي الثاني

$$d_2(x, y) \text{ ومتساويين للبعض همها وكلاهما يساوي } |x - y|$$

• نفرض ان $\forall \epsilon > 0$ لكي نجد $\delta > 0$ بحيث ان

$$d_1(x, a) < \delta \text{ وان } d_2(f(x), L) < \epsilon$$

• تكون القيم داخل المطلق متساوية للتطبيقين بعد سحب عامل

$$\text{مشترك ع الاغلب من } d_2(f(x), L)$$

• هذا المشترك نتخلص منه بفرض ان δ تساوي أي قيمة يمكننا

التخلص من ذلك الثابت مع وجود ϵ

• اذا كان $d_2(f(x), L) < \epsilon$ فان $\lim_{x \rightarrow a} f(x) = L$

Theorem: if f has a limit point L , then this limit is unique.

Properties of limits of functions:

Let $S \subseteq R^m$, $f: S \rightarrow R^m$, and $g: S \rightarrow R^m$; s.t

$\lim_{x \rightarrow a} f(x) = A$, $\lim_{x \rightarrow a} g(x) = B$, then

$$1. \quad \lim_{x \rightarrow a} (f + g)(x) = A + B$$

$$2. \quad \lim_{x \rightarrow a} (f \cdot g) = A B$$

$$3. \quad \lim_{x \rightarrow a} \left(\frac{f}{g} \right) = \frac{A}{B} \quad \text{but } B \neq 0, g(x) \neq 0$$

Def: Let (X, d_1) and (Y, d_2) are two metric spaces
suppose that $S \subseteq X$, $p \in S$ and $f: S \rightarrow Y$ we say **f**
continuous at $x = p$ **iff** $\forall \epsilon > 0, \exists \delta > 0$ s.t
 $d_2(f(x), f(p)) < \epsilon$ whenever $d_1(x, p) < \delta$. If **f** is
continuous at $x, \forall x \in S$, then **f** is continuous on S .

Example: prove that $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$ is
continuous, $\forall x \in X$,
 $X = Y = \mathbb{R}, d_1 = d_2 = |x - y|$?

Solution:

Let $p \in \mathbb{R}$ to prove **f** continuous at $x = p$?

Let $\epsilon > 0$, to find $\delta > 0$ s.t $|f(x) - f(p)| < \epsilon$,
if $|x - p| < \delta$.

$$|f(x) - f(p)| = |x^2 - p^2| = |(x - p)(x + p)| = |x - p||x + p|.$$

$$\leq |x - p| (|x| + |p|) \dots\dots\dots(1)$$

Since: $||x| - |p|| \leq |x - p| < \delta, \rightarrow$

$$||x| - |p|| < \delta.$$

$$\rightarrow -\delta < |x| - |p| < \delta \rightarrow |x| - |p| < \delta \rightarrow$$

$$|x| < \delta + |p| \dots\dots\dots(2)$$

Put (2) in (1)

$$|f(x) - f(p)| \leq |x - p|(|x| + |p|) < |p| + \delta + |p|$$

$$< \delta(2|p| + \delta) \leq \delta(2|p| + 1)$$

(since $0 < \delta \leq 1$) .choose $\delta = \min \left\{ 1, \frac{\epsilon}{2|p|+1} \right\}$

$$\rightarrow |f(x) - f(p)| < \frac{\epsilon}{2|p|+1} \cdot (2|p| + 1) = \epsilon$$

$\therefore f$ is continuous at $x = p, \forall p \in R, \rightarrow f$ is continuous on R

كيف نبرهن عن استمرارية الدالة:

• الهدف هو الوصول الى $|f(x) - f(p)| < \epsilon$

• نفرض ان

Let $p \in R$ T.P f continuous at $x = p$

• ونفرض أيضا

$$\epsilon > 0 \text{ to find } \delta > 0 \text{ s.t. } |f(x) - f(p)| < \epsilon \text{ if } |x - p| < \delta$$

- نحل المعادلة $|f(x) - f(p)|$ وباستخدام خواص المطلق نحصل معادلة رقم 1
- ومن خواص المطلق القيمة بين الموجب والسالب وبعزل x في طرف وبقية المتغيرات في طرف اخر نحصل على معادلة 2
- نأخذ min بين العدد الحقيقي ϵ على (مقام) القيمة المشتركة بين الطرفين من تعويض 2 في 1

Theorem: Let (X, d_1) and (Y, d_2) are two metric spaces $f: X \rightarrow Y$ be a function. Then f is continuous at $p \in X$ iff $\lim_{n \rightarrow \infty} f(x_n) = f(p)$ for each sequence $\langle X_n \rangle$ in X and $X_n \rightarrow p$.

Example:

1- Let $f: \mathbb{R} \rightarrow \mathbb{R}$ s.t $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$ use

theorem to prove f is not continuous at $p = 0$

Sol:

Let $\langle X_n \rangle = \langle \frac{1}{n} \rangle$ then $\frac{1}{n} \rightarrow 0$ as $n \rightarrow$

$\infty \forall n, f(x) = f\left(\frac{1}{n}\right) = 1$, then

$f(x_n) = f\left(\frac{1}{n}\right) = 1, 1, \dots$ converge to

$1 \neq 0 = f(0)$

by theorem above (...), f is not continuous at $p = 0$.

2- Let $f: [a, b] \rightarrow \mathbb{R}$, s.t $f(x) = \begin{cases} 1 & \text{if } x \in Q \\ 2 & \text{if } x \in Q' \end{cases}$ use

theorem above to prove that f is not continuous

Sol:

Let $p \in [a, b] \rightarrow p \in Q$ or $p \in Q'$

i. If $p \in Q$ since (between any two reals there are infinitely rationales and irrationals)

And (for every real p , there is irrational (or rational) Cauchy seq. Converge to p)

\Rightarrow there is an irrational Sequence $\langle X_n \rangle$

$x_n \rightarrow p$.

$\Rightarrow f(x_n) = 2$, there $\langle f(x_n) \rangle = 2, 2, 2, \dots$,

Converge to $2 \neq 1 = f(p)$

by theorem above: (...), f is not continuous at $p \forall p \in Q$.

ii. If $p \in Q'$,

by similar way we can show that f is not continuous at $p, \forall p$ s.t $p \in Q'$,

$\therefore f$ is not Continuous at $p, \forall p \in [a, b]$

Lecher 2:

Theorem: Let (X, d_1) and (Y, d_2) are two metric spaces, $f: X \rightarrow Y$ be a function, then f is continuous on X iff $f^{-1}(V)$ is open set in $X, \forall v$ is opens set in Y .

Example: Let $f: R \rightarrow R$ s.t $f(x) = x^2$. Use theorem above to prove that f is Continuous on R ?

Proof:

Let V open set in R , then $V = (a, b)$ there are three case

1. If $a, b > 0 \rightarrow f^{-1}(V) = (\sqrt{a}, \sqrt{b}) \cup (-\sqrt{b}, -\sqrt{a})$ open

$\therefore f^{-1}(V)$ is open in X

2. If $a < 0, b > 0 \rightarrow f^{-1}(V) = (-\sqrt{b}, \sqrt{b})$ open

$\therefore f^{-1}(V)$ is open in X

3. If $a, b < 0 \rightarrow f^{-1}(V) = \emptyset$ open $\rightarrow f^{-1}(V)$ is open in X

$\therefore \forall V$ open in $Y \rightarrow f^{-1}(V)$ is open in $X \rightarrow f$ is continuous on X by theorem: (...)

Example: Let $f: R \rightarrow R$ s.t $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$.

Use theorem above to prove that f is not continuous at 0?

Proof:

Let $V = \left(-\frac{1}{2}, \frac{1}{2}\right)$ is open set in Y

Then $f^{-1}(V) = \{0\}$ is closed set in X

by theorem above f is not continuous at $p = 0$

Theorem: Let (X, d) be a metric space and f, g are two real valued functions, if f, g are continuous, then

1. $f \pm g$
2. $f \cdot g$
3. $rf, r \in \mathbb{R}$
4. $\frac{f}{g}, g \neq 0$ are continuous

Corollary: every polynomial $p(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$ is continuous where a_i is constant.

Proof:

Since $f(x) = x^n$ is a continuous function, $\forall n$ and rf is continuous by theorem, then the **sum of continuous functions is continuous** by theorem above.

The Properties of Real Valued Functions Defined on Compact Spaces

Theorem: Let (X, d_1) and (Y, d_2) be two metric spaces and $f: X \rightarrow Y$ a continuous function, if X is compact, then $f(X)$ is compact.

Proof:

$f(x) = \{ f(x) : x \in X \}$, Let $\{V_\lambda : \lambda \in \Lambda\}$ be an open set in Y , $\forall \lambda \in \Lambda$ and f is Continuous then $f^{-1}(V_\lambda)$ is open set in X , $\forall \lambda \in \Lambda$ (by theorem)

Since $f(x) \subseteq \bigcup_{\lambda \in \Lambda} V_\lambda$

$\rightarrow X \subseteq f^{-1}(\bigcup_{\lambda \in \Lambda} V_\lambda) = \bigcup_{\lambda \in \Lambda} (f^{-1}(V_\lambda))$

$\rightarrow \{f^{-1}(V_\lambda) : \lambda \in \Lambda\}$ is open cover for X .

\rightarrow there is finite subcover

$\{f^{-1}(V_1), f^{-1}(V_2), \dots, f^{-1}(V_n)\}$ for X (since X is compact)

$\rightarrow X \subseteq \bigcup_{i=1}^n f^{-1}(V_i) = f^{-1}(\bigcup_{i=1}^n V_i)$

$\rightarrow X \subseteq f^{-1}(\bigcup_{i=1}^n V_i)$

$\rightarrow f(x) \subseteq \bigcup_{i=1}^n V_i$

$\rightarrow \{V_1, V_2, \dots, V_n\}$ is finite subcover for $f(x)$

$\therefore f(x)$ is compact

Lecher 3:

Definition: Let (X, d) be a metric space and $f: X \rightarrow R$ we say that f is bounded if $\exists M > 0$, s.t $|f(x)| \leq M, \forall x \in X$.

i.e.: $f(x) = \{f(x) : x \in X\}$ is bounded set if has upper and lower bonded

i.e.: f is bounded iff $f(x)$ is bounded set.

Theorem: Let (X, d) be a metric space and $f: X \rightarrow \mathbb{R}$ continuous function, if X is compact, then f is bounded.

Proof:

Since f is continuous and X is compact

Then $f(x)$ is compact by theorem.

$\rightarrow f(x)$ is closed and bounded.

$\rightarrow f$ is bounded.

Example: Give example for bounded function and its domain not compact.

Sol:

$X = (0,1)$ not compact, $f: X \rightarrow \mathbb{R}$, $f(x) = 3x$ if $M = 3 \rightarrow |f(x)| \leq 3$, then f is bounded.

Remark: the condition of compactness in above theorem is necessary.

Example: Let $X = (0, \infty)$ not compact, $f(x) = \frac{1}{x}$ continuous, $X > c$ f is not bounded since

$$\forall M > 0, \exists k \in \mathbb{N} \text{ s.t. } M < \frac{1}{k} = f(x)$$

$$\forall x \in X, \exists M > 0 \text{ s.t. } M|f(x)| \leq M$$

(by Arch. Prop.)

Definition: Let (X, d) be a metric space and $f: X \rightarrow R$ bounded function, a point a is called.

1. Maximum extreme point of f if

$$f(x) \leq f(a), \forall x \in X$$

2. Minimum extreme point of f if

$$f(a) \leq f(x), \forall x \in X$$

Theorem: Let $f: X \rightarrow R$ be a continuous function, if X is compact, then $\exists a, b \in X$, such that $f(a) \leq f(x) \leq f(b), \forall x \in X$

i.e.: f has min. extreme point at a , and f has max. extreme point at b .

When you see **Compact** remember

1. Domain function is **closed** interval
2. If **X** is compact then $f(x)$ is compact
3. If **X** is compact then f is **bounded**
4. If $f(x)$ is compact then $f(x)$ is **closed** and **bounded**

Lecher 4:

Uniformly continuity

Definition: Let (X, d) be a metric space and $f: X \rightarrow R$, f is called uniformly continuity if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |f(x) - f(y)| < \epsilon \text{ whenever } d(x, y) < \delta, \forall x, y \in X$$

Remark: The choose of δ in the definition of uniformly continuous is depending on ϵ only

Theorem: every uniformly continuous is continuous.

Proof:

Let f be a uniformly continuous on X , then

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |f(x) - f(y)| < \epsilon$$

whenever $d(x, y) < \delta, \forall x, y \in X$

\rightarrow take $y = p \rightarrow \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |f(x) - f(p)| < \epsilon$ whenever

$d(x, p) < \delta, \forall x, p \in X$, then f is continuous at $p, \forall p$.

Remark: The converse of theorem above is not true for example:

Example: Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$ is continuous function let $x = n$, $y = n + \frac{1}{n}$, $n \in \mathbb{N}$

$$d(x, y) = |x - y| = \left| n - n - \frac{1}{n} \right| = \left| -\frac{1}{n} \right| = \frac{1}{n} < \delta.$$

By Arch. Prop. Any real δ , $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \delta$.

$$\begin{aligned} \text{Take } \epsilon = 1 \rightarrow |f(x) - f(y)| &= \left| n^2 - \left(n + \frac{1}{n} \right)^2 \right| \\ &= \left| n^2 - n^2 - 2 - \frac{1}{n^2} \right| \\ &= 2 + \frac{1}{n^2} > \epsilon = 1 \end{aligned}$$

$\therefore f$ is not uniformly continuous.

Theorem: Let $f: X \rightarrow \mathbb{R}$ be a continuous function. If X is compact, then f is uniformly continuous.

Example: Let $f: [a, b] \rightarrow \mathbb{R}$, $f(x) = 3x$ is continuous then f is uniformly continuous.

Any closed interval
 $A = [a, b]$ is Compact

Theorem: (Intermediate Value property) (I.V.P)

Let f be a continuous on $[a, b]$ and

$f(a) = \alpha, f(b) = \beta$, then for all

$\gamma, \alpha < \gamma < \beta$,

$\exists c, a < c < b$ and $f(c) = \gamma$

Proof:

Let $S = \{x: x \in [a, b], f(x) \leq \gamma\}$.

$S \neq \emptyset$, Since $a \in S$, S is bounded above

(Since b is upper bounded of S)

\rightarrow by Completeness axiom, S has sup. Let

$\sup(S) = c$.

Then there are three cases $f(c) =$

γ or $f(c) < \gamma$ or $f(c) > \gamma$

If $f(c) < \gamma$:

Since f is continuous at c and

$f(c) < \gamma \rightarrow \exists \epsilon > 0$

s.t $f(x) < \gamma, \forall x \in (c - \epsilon, c + \epsilon)$

If $c < x < b \rightarrow x \in S$ and $c < x \rightarrow !$

$\rightarrow f(c) \neq \gamma$ (since $\sup(S) = c$)

If $f(c) > \gamma$, by Similar way $f(c) \neq \gamma$

$\rightarrow f(c) = \gamma \rightarrow \exists c, a < c < b$ and

$f(c) = \gamma$

Theorem: (**Interval theorem**)

If f is continuous on $I = [a, b]$, then $f(I)$ is closed bounded interval.

Proof:

Since I is closed and bounded, then I is compact.

Since f is continuous on I then f has max, min extreme point $\rightarrow \exists c, d, f(c) = m, f(d) = M$ s.t $m \leq f(x) \leq M, \forall x$

There are two cases: $c < d$ or $d < c$

If $c < d$, Apply (I.V.P) on f and $[c, d]$

$\rightarrow \forall y, y \in (m, M), \exists x \in (c, d)$ s.t $f(x) = y$

$\therefore f(I) = [m, M]$

Theorem: (**Fixed point theorem**)

Let $f: [0,1] \rightarrow [0,1]$ continuous, then there is at least one number c (c is called fixed point) such that $f(c) = c$

Proof:

Suppose $g: [0,1] \rightarrow \mathbb{R}$, s.t $g(x) = f(x) - x$ continuous on $[0,1]$ (since the identity function $c: X \rightarrow X$ is continuous and f continuous, then g is continuous on $[0,1]$).

If $f(0) = 0$ or $f(1) = 1$, then proof is complete.

Suppose $f(0) \neq 0$ and $f(1) \neq 1$

Since f is onto ($f: [0,1] \rightarrow [0,1]$) $\rightarrow g(0) = f(0) - 0 = f(0) > 0$

and $g(1) = f(1) - 1 < 0 \rightarrow g(1) < 0 < g(0)$

by (I.V.P), $\exists c, 0 < c < 1$ s.t $g(c) = 0 \rightarrow$

$f(c) - c = 0 \rightarrow f(c) = c$

$\therefore f$ has a fixed point

Theorem: suppose $f: I \rightarrow J$ is bijective, where I, J are closed interval, if f is continuous, then f^{-1} continuous

Theorem: Any polynomial of odd degree has at least one real root.

Lecher 5:

Chapter Six

The Differentiation

Definition: A function $f: A \rightarrow Y$, $A \subseteq R$, is called differentiable at c iff $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ exists.

The value of this limit is called the derivative of f at c and denoted by $f'(c)$

Theorem: if f is differentiable at point c , then it is continuous at c

Proof:

Let $\epsilon > 0$, T.P $|f(x) - f(c)| <$

Since f is diff., then $f'(c) = \lim$

$\lim_{x \rightarrow c} (f(x) - f(c)) = \lim_{x \rightarrow c}$

$f(c) \frac{x-c}{x-c}$

$$\begin{aligned} &= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \rightarrow c} (x - c) \\ &= f'(c) = 0 \end{aligned}$$

Where came this $f(x) - f(c) \frac{x-c}{x-c}$

We know that def. of Continuous is $|f(x) - f(c)|$ and in same time we know that def. of diff. is

$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ by put the tow def.

$|f(x) - f(c)| = \frac{f(x) - f(c)}{x - c}$ but not equal

we can be it is equal by multiplication $\frac{x-c}{x-c}$ on the right side. It is will be as

$f(x) - f(c) = f(x) - f(c) \frac{x-c}{x-c}$

After that take limit, both said

So, $\lim_{x \rightarrow c} (f(x) - f(c)) = 0$, then

$$|f(x) - f(c)| < \epsilon$$

Hence f is continuous at c .

Remark: The converse of the above theorem is not true

Example: Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = |x|$ continuous, but not diff at $c = 0$

$$\text{Since, } f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{|x| - 0}{x - 0} = \lim_{x \rightarrow 0} \frac{|x|}{x} = \pm \frac{x}{x} = \pm 1, \text{ not}$$

exists

Hence $f'(0)$ is not exists

So, f is not diff at $c = 0$

Limit value is **unique** except that does not exist

Theorem: (Rules of derivative)

Let f, g be two diff function at c , then

1. $(f + g)$ is diff at c and $(f + g)'(c) = f'(c) + g'(c)$

2. $(f \cdot g)$ is diff at c and $(f \cdot g)'(c) = f(c)g'(c) + g(c)f'(c)$

3. If $f(c) \neq 0$, then $\frac{1}{f(c)}$ is diff at c and

$$\left(\frac{1}{f}\right)'(c) = \frac{-f'(c)}{(f(c))^2}$$

4. If $g(c) \neq 0$, then $\frac{f}{g}$ is diff at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{g \cdot f' - f \cdot g'}{(g(c))^2}$$

5. If, $\forall n \in \mathbb{N}$, $f(x) = x^n$, then $f'(x) = n x^{n-1}$

Theorem: (Chain Rule)

Let f be diff function at c and g diff at $b = f(c)$ then $g \circ f$ is diff at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

Theorem: (Invers function theorem)

Suppose that $f: A \rightarrow B$ is bijective function, where A and B are intervals. If f is diff. at $a \in A$ and $f'(a) \neq 0$ then f^{-1} is diff at $b = f(a)$ and

$$(f^{-1})' = \frac{1}{f'(a)}$$

Remember: not any function has Invers. To be has Invers it must be bijective (تقابل). That is mean the function is (1-1 and onto)

1-1 is every element in Doman joint in one element in codomain

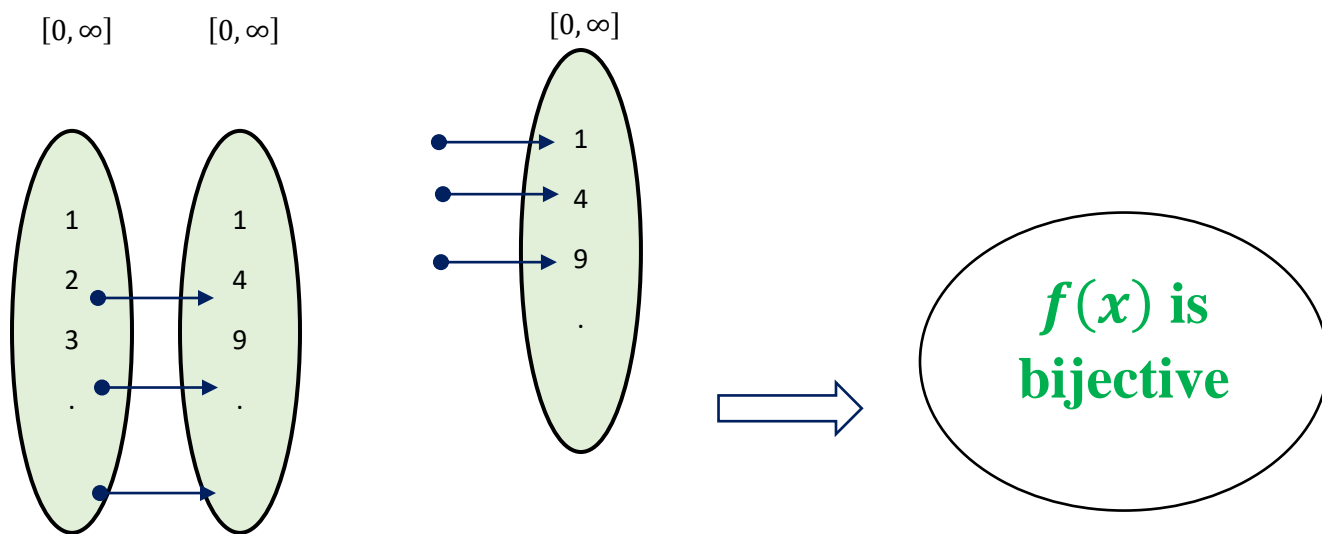
Onto is every element in codomain joint

Example: Let $f: [0, \infty) \rightarrow [0, \infty)$ s.t $f(x) = x^2$? f is 1 - 1 and onto, then f^{-1} exists.

Solution:

$$\because y = x^2, \rightarrow x = \sqrt{y}, f: [0, \infty) \rightarrow [0, \infty) \text{ and } f^{-1}(y) = \sqrt{y}.$$

$$\forall y, (f^{-1}(y))' = \frac{1}{f'(x)} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}.$$



Condition of 1-1

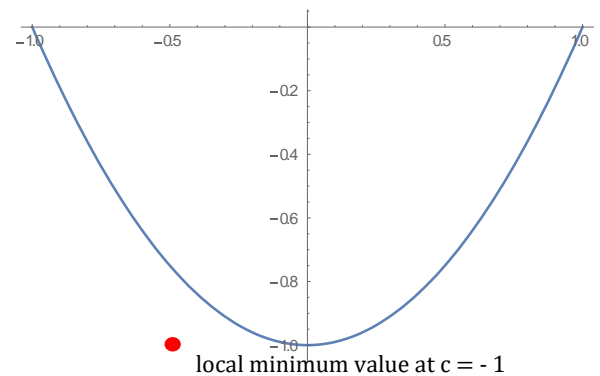
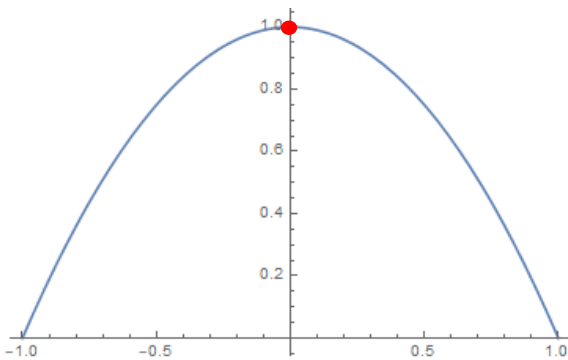
Condition of onto

Remember: f has Invers **iff** f is bijective

Definition: A function f has a local maximum value at c , if $\exists J$ (J is open interval), $C \in J$, s.t $f(x) \leq f(c), \forall x \in J$.

A function f has a local minimum value, if $\exists J$ (J is open interval), $C \in J$ s.t $f(c) \leq f(x), \forall x \in J$.

local maximum value at $c = 1$



Theorem: (Local Extreme Value)

If f is diff. at c and has local Max. (Or min.) value at c , then $f'(c) = 0$

Proof:

Suppose f has min. value at $c \rightarrow$ by def. $\exists J$ (J is open interval),

$C \in J, s. t f(c) \leq f(x), \forall x \in J$.

Then $f(x) - f(c) \geq 0$. there are two cases:

$$1. \quad \text{If } x > c \rightarrow x - c > 0 \Rightarrow \frac{f(x) - f(c)}{x - c} \geq 0 \rightarrow$$

$$f'_t(c) = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} > 0$$

$$2. \quad \text{If } x < c \rightarrow x - c < 0 \Rightarrow \frac{f(x) - f(c)}{x - c} \leq 0 \rightarrow$$

$$f'_t(c) = \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \leq 0$$

Since f is diff at c , then $0 \leq f'_+(c) = f'(c) = f'_-(c) \leq 0 \rightarrow f'(c) = 0$

Local maximum or minimum

1. The function must be diff.
2. Either has local maximum Or has local minimum
3. First Derivative when equal zero solve this equation to find the local point
4. Put the local point in Derivative to know type of local it must be either greater than 0 or less than zero
5. When you put local point and have got 0 find the next derivative to get negative or positive number
6. Positive $f''(c) > 0$ is local Minimum point or negative $f''(c) < 0$ Local maximum

Lecher 6:

Remark: The converse of the above theorem is not true

Example: Let $f: \mathbb{R} \rightarrow \mathbb{R}$, s.t $f(x) = x^3$ diff on \mathbb{R}
 $f'(x) = 3x^2 \rightarrow x = 0$

Theorem: (Roll's theorem)

If f is diff on (a, b) and Continuous $[a, b]$ and $f(a) = f(b)$, then there exists c , $a < c < b$, such that $f'(c) = 0$

Proof:

Since $[a, b]$ is compact and f cont. on $[a, b]$, then f has max. and min $\rightarrow \exists c_1, c_2 \in [a, b]$ s.t $f(c_1) = \max$. Value $f(c_2)$ min. Value.

Now, if $f(c_1) \neq f(c_2) \Rightarrow$ at least one of c_1 and c_2 is not equal to a or $b \rightarrow f$ has a local max. or min (or both) in $[a, b]$.

By theorem (...), $f'(c_1)$ or $f'(c_2)$ (or both) equal to zero

Example: find the value of c , which satisfy Roll's theorem,

where $f: [-1,1] \rightarrow R$, s.t

$$f(x) = \sqrt{1 - x^2}.$$

Solution:

$$1 - x^2 \geq 0 \rightarrow -x^2 \geq -1 \rightarrow$$

$$x^2 \leq 1 \rightarrow -1 \leq x \leq 1$$

$\therefore f$ is diff on $(-1, 1)$ and continuous on $[-1, 1]$

$a = -1, b = 1$, then

$$f(-1) = 0 = f(1)$$

by Roll's theorem,

$$\exists c, -1 < c <$$

$$1 \text{ s.t } f'(c) = 0$$

$$\therefore f'(x) = -\frac{2x}{2\sqrt{1-x^2}} \text{ and}$$

$$f'(c) = 0 \rightarrow c = 0$$

Find c satisfy Roll's theorem

1. Find Rang of function by differential rules
2. Make sure that the function is differentiable and continuous over the given interval
3. $f(a) = f(b)$
4. by Roll's theorem
 $\exists c \in a \leq c \leq b$
s.t $f'(c) = 0$
5. find $f'(x)$
6. change all x by c in $f'(x) \rightarrow f'(c)$
7. now $f'(c) = 0$ to find c

Theorem: (Mean Value theorem) (M.V.T)

If f is diff. on (a, b) and continuous on $[a, b]$, then there exists $c, a < c < b$, such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof:

Define $g(x) = f(x) - \lambda x$, $\lambda = \frac{f(b)-f(a)}{b-a}$

Since f continuous on $[a, b]$ and λx continuous on $[a, b]$, then g is continuous on $[a, b]$

Also, f diff on (a, b) and λx diff. on (a, b) , there g is diff. on (a, b) .

$g(a) = g(b) \rightarrow \exists c \in (a, b)$ s.t $g'(c) = 0$

Since $g'(x) = f'(x) - \lambda \rightarrow g'(c) = f'(c) - \lambda = 0$

$$\rightarrow f'(c) = \lambda, \text{ then } f'(c) = \frac{f(b)-f(a)}{b-a}$$

Example: Let $f: [0,1] \rightarrow R$, $f(x) = \sqrt{x}$, find the Value of c which satisfy M.V.T.

Solution:

f is continuous on $[0,1]$ and diff. on $(0,1)$ and

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

$$\frac{f(b)-f(a)}{b-a} = \frac{1-0}{1-0} = 1 \Rightarrow f'(c) = \frac{1}{2\sqrt{c}} = 1 \rightarrow$$

$$\frac{1}{\sqrt{c}} = 2$$

$$c = \frac{1}{4}$$

Corollary: if f is continuous on $[a, b]$ and diff. on (a, b) , $f'(x) = 0, \forall x \in (a, b)$, then f is constant.

Proof:

Let $x_1, x_2 \in [a, b]$ and $x_1 < x_2$, since f is continuous

Since f is continuous on $[a, b]$, then f continuous on $[x_1, x_2]$ and f is diff. on (a, b) ,
So, f is diff on (x_1, x_2) .

by M.V.T, $\exists c, c \in (x_1, x_2)$ s.t $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$

so $f(x_2) - f(x_1) = 0 \Rightarrow f(x_2) = f(x_1)$ for any x_1, x_2 in $[a, b]$, then f is constant function.

Corollary: if f continuous on $[a, b]$, diff. on (a, b) and $f'(x) \geq 0, \forall x \in (a, b)$, then f is an increasing [if f is an increasing if $x_1 < x_2$, then $f(x_1) \leq f(x_2)$]

Corollary: if f continuous on $[a, b]$, diff. on (a, b) and $f'(x) \leq 0, \forall x \in (a, b)$, then f is an decreasing [f is an decreasing if $x_1 < x_2$, then $f(x_2) \leq f(x_1)$]

Definition: if f is n -time diff. and $f^{(n)}$ is continuous function then f is called n -times continuously differentiable.

Taylor's theorem: suppose $\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of Convergence R , then $|x| < R$.
i.e.: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $f'(x) = \sum n a_n x^{n-1}, |x| < R$.

Lecher 7:

Definition: let f be n -times continuous diff. Function at 0, the Taylor polynomial of degree n for f at 0 $\left(f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots + \frac{f^n(0)}{n!} x^n \right)$.

Is definition by $T_n \left(f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots + \frac{f^n(0)}{n!} x^n \right)$

Example: let $f(x) = e^x$, $f^{(r)}(x) = e^x$, $r = 1, 2, \dots$

$$f^{(n)}(0) = 1, \forall r.$$

$$T_0 f(x) = 1$$

$$T_1 f(x) = f(0) + \frac{f'(0)}{1!} x = 1 + x$$

$$T_2 f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 = 1 + x +$$

$$\frac{x^2}{2}$$

نلاحظ ان متعددة حدود تايلور تقترب من الدالة e^x بالقرب

من النقطة $x = 0$

ساكنة

Definition: the point x is called stationary if

$$f'(0) = 0$$

Theorem: Classification theorem for local extrema

If f is $(n+1)$ -times continuously diff on open interval contains a and $f^{(r)}(a) = 0$, $r = 1, 2, \dots, n$ and $f^{(n+1)}(a) \neq 0$, then

1. $n+1$ even, $f^{(n+1)}(a) > 0$, $\rightarrow x = a$, local min. of f .
2. $n+1$ even, $f^{(n+1)}(a) < 0$, $\rightarrow x = a$, local max. of f .
3. $n+1$ odd, f has neither a local max. n or local min. at $x = a$

Example: determine the nature of the stationary point of $f(x)$ where $f(x) = x^6 - 6x^4$

Solution:

Since f is polynomial function, then f infinitely diff.

$$\begin{aligned} f'(x) &= 6x^5 - 24x^3 \\ &= 6x^3(x^2 - 4) = 6x^3(x - 2)(x + 2) = \\ &0 \end{aligned}$$

$\therefore f$ has stationary point at $x = 0, x = 2, x = -2$

$$f''(x) = 30x^4 - 72x^2$$

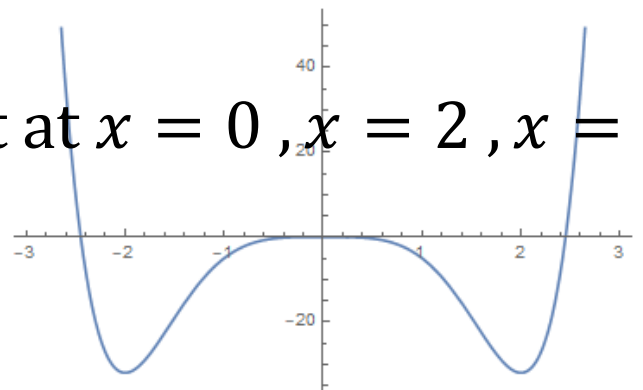
$$f''(\pm 2) = 192 > 0$$

So f has a local min at $x = \pm 2$

$$f^{(3)}(x) = 120x^3 - 144x$$

$$f^{(3)}(0) = 0$$

$$\text{So } f^{(4)} = 360x^2 - 144$$



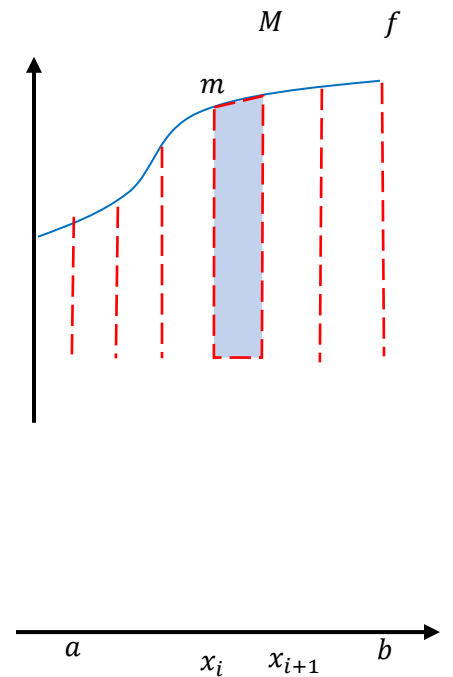
$f^{(4)}(0) = -144 < 0$
 The f has a local max at $x = 0$

Chapter Seven

Reiman Integral

تكامل ريمان

ان فكرة ريمان هو حساب المساحة للمنطقة



.....

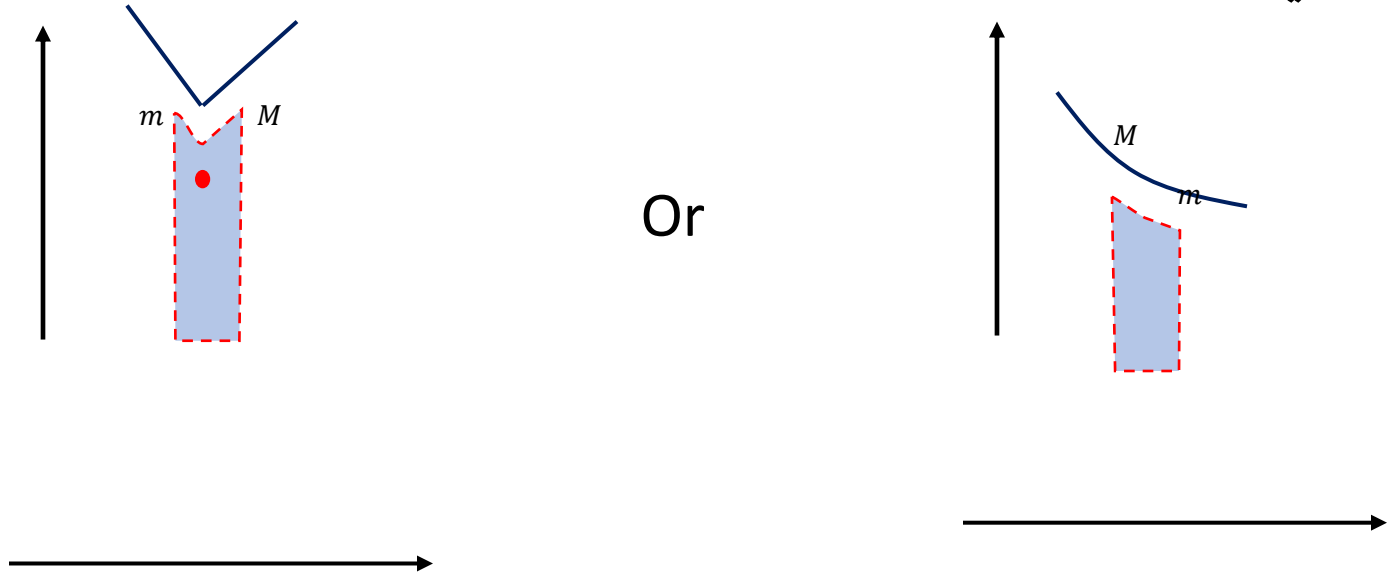
$$\left\{ \begin{array}{l} x_{i+1} - x_i = \text{العرض} \\ m_i = f(x_i) = \text{الطول} \end{array} \right.$$

)

$$\left\{ \begin{array}{l} x_{i+1} - x_i = \text{العرض} \\ M_i = f(x_{i+1}) = \text{الطول} \end{array} \right.$$

فان مجموع المساحات في (1) يسمى بمجموع ريمان الأدنى.
 ومجموع المساحات في (2) يسمى بمجموع ريمان الأعلى اما اذا

كانت المستطيلات في التجزئة انعم فان المجموع (1) يقترب من المجموع (2). مع ملاحظة انه قد لا تكون اعلى قيمة او اقل قيمة للدالة هي صورة احدى رؤوس الفترة $[x_i, x_{i+1}]$ كما في الشكل التالي:



Reiman Integrable:

Let f be a bounded function defined on $[a, b] = J$, where $|J| = b - a$ is the **length** of J . if

$\Pi = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ is **ordered** subset of J where the first element is **a** and the last is **b**, then Π is called the **partion** of J

And $J_i = [x_i, x_{i+1}]$, $i = 0, 1, 2, \dots, n - 1$ are called the subintervals of J .

$$\Pi: x_0 < x_1 < x_2 < \dots < x_n$$

Since f is bounded function, So

To Make Partion for Interval

1. Make set as $\Pi = \{a, \dots, b\}$
2. a : is first (Start Interval)
3. b : is second (End Interval)
4. Numbers between a and b are Order

$$M = \sup\{f(x): x \in J\} , M_i = \sup\{f(x): x \in J_i\}$$

$$m = \inf\{f(x): x \in J\} , m_i = \inf\{f(x): x \in J_i\}$$

- Lower sum of f relative to Π .

$$\underline{R}(f, \Pi) = \sum_{i=0}^{n-1} m_i |J_i|$$

$$= \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i)$$

Don't forget
 J : is Main interval
 J_i : is Subinterval

- Upper sum of f relative to Π

$$\overline{R}(f, \Pi) = \sum_{i=0}^{n-1} M_i |J_i|$$

$$= \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i)$$

Lecher 8:

Remark: $\forall i , m \leq m_i \leq M_i \leq M$

Proof:

$$\text{If } A = \{f(x): x \in J\}$$

$$B = \{f(x) : x \in J_i\} \implies B \subseteq A$$

By So, $\inf(A) \leq \inf(B)$ (properties of inf, sup).

$$\Rightarrow m \leq m_i, \forall i$$

And $\sup(B) \leq \sup(A)$, then M

Also, Since $\inf(B) \leq \sup(B)$

$$M_i \leq M.$$

Remember
 f is bounded iff $f(x)$ is bounded
 (Has Upper & Lower bound)

Remark: if f is bounded function defined on $[a, b]$, then

$$m(b - a) \leq \underline{R}(f, \Pi) \leq \bar{R}(f, \Pi) \leq M(b - a)$$

Proof:

Since $m \leq m_i \leq M_i \leq M$ (by above Remark).

$$\Rightarrow m |J_i| \leq m_i |J_i| \leq M_i |J_i| \leq M |J_i|, \forall i$$

$$\Rightarrow \sum_{i=0}^{n-1} m |J_i| \leq \sum_{i=0}^{n-1} m_i |J_i| \leq \sum_{i=0}^{n-1} M_i |J_i| \leq \sum_{i=0}^{n-1} M |J_i|$$

$$\Rightarrow m \sum_{i=0}^{n-1} |J_i| \leq \underline{R}(f, \Pi) \leq \bar{R}(f, \Pi) \leq M \sum_{i=0}^{n-1} |J_i|$$

$$\Rightarrow m(b - a) \leq \underline{R}(f, \Pi) \leq \bar{R}(f, \Pi) \leq M(b - a)$$

Definition: Let Π, Π^* be two partion of $[a, b]$, if $\Pi \subseteq \Pi^*$ then Π^* is called the refinement of Π .

Example: Let $\Pi = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ be a partion of $[0, 1]$

then $\Pi^* = \left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\}$ also partion of $[0, 1]$

and $\Pi \subseteq \Pi^*$

Remark: if f is bounded function on $[a, b]$ and Π^* is refinement of Π , then.

1. $\underline{R}(f, \Pi) \leq \underline{R}(f, \Pi^*)$
2. $\overline{R}(f, \Pi^*) \leq \overline{R}(f, \Pi)$
3. $\underline{R}(f, \Pi) \leq \overline{R}(f, \Pi^*)$

Now, $\overline{R}(f) = \{ \overline{R}(f, \Pi) : \Pi \text{ is partition} \} \neq \phi$
Since \exists a partition which is $[a, b]$ itself.

And, $\underline{R}(f) = \{ \underline{R}(f, \Pi) : \Pi \text{ is partition} \} \neq \phi$

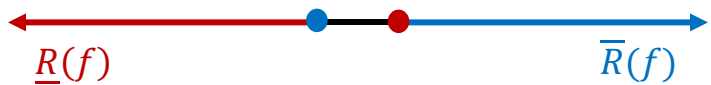
So, there exist at least one $\underline{R}(f, \Pi)$ and at least one $\overline{R}(f, \Pi)$

Also, since $\underline{R}(f, \Pi) \leq \overline{R}(f, \Pi^*)$, there any element in \overline{R} will be upper bounded of \underline{R} . And any element in \underline{R} will be lower bounded of $\overline{R}(f)$. So

$$\int f = R \int f = \inf(\overline{R}, f) \quad f \overset{\text{sup}}{\text{—}} \text{تکامل ریمان الأعلى}$$

$$\int f = R \int f = \sup(\underline{R}, f) \quad f \overset{\text{inf}}{\text{—}} \text{تکامل ریمان الأدنى}$$

$$\int f \leq \int f$$



Definition: Let f be a bounded function definition on $[a, b]$ f is Reiman integrable iff

$$\underline{\int} f = \overline{\int} f, \text{ and denoted by } R \int f \text{ or } \int_a^b f.$$

Example: find $\int_0^2 f(x)$, s. t $f(x) = 3x$.

Solution:

$$[a, b] = [0, 2] = J$$

$$|J| = b - a = 2 - 0 = 2.$$

$$\Pi = \left\{ a < a + \frac{b-a}{n} < a + 2 \frac{b-a}{n} < \dots < a + n \frac{b-a}{n} = b \right\}$$

i.e.: $0 < \frac{2}{n} < \frac{4}{n} < \frac{6}{n} < \frac{8}{n} < \dots < 2$

$$\left\{ 0 < 0 + \frac{2+0}{n} < 0 + 2 \frac{2}{n} < \dots < 0 + n \frac{2}{n} = b \right\}$$

$$0 < \frac{2}{n} < 2 \cdot \frac{2}{n} < \dots < 2$$

$$0 < \frac{2}{n} < \frac{4}{n} < \dots < 2$$

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$x_0 \quad x_1 \quad x_2 \quad x_n$$

Remember

$$|J_i| = b_i - a_i$$

$$\text{as: } 2 - \frac{2(n-1)}{n} = \frac{2n-2n+2}{n} = \frac{2}{n}$$

The sub intervals are $J_0 = \left[0, \frac{2}{n}\right]$

$$J_1 = \left[\frac{2}{n}, \frac{4}{n}\right]$$

$$J_2 = \left[\frac{4}{n}, \frac{6}{n}\right]$$

⋮

$$J_{n-1} = \left[\frac{2(n-1)}{n}, 2\right]$$

$$|J_i| = \frac{2}{n}, \forall i.$$

الآن نجد

المجموع من الأدنى

$$\underline{R}(f, \Pi) = \sum_{i=0}^{n-1} m_i |J_i|$$

$$= m_0(x_1 - x_0)$$

$$+ m_1(x_2 - x_1) + \dots$$

$$+ m_{n-1}(x_n - x_{n-1})$$

$$= f(x_0) \frac{2}{n} + f(x_1) \frac{2}{n} + \dots +$$

$$f(x_{n-1}) \cdot \frac{2}{n}$$

بما ان الدالة متزايدة فان صورة بداية

الفترة دائما هي اقل الصور

$$= 0 \cdot \frac{2}{n} + \frac{6}{n} \cdot \frac{2}{n} + \frac{12}{n} \cdot \frac{2}{n} + \dots +$$

$$\frac{6(n-1)}{n} \cdot \frac{2}{n}$$

$$\begin{aligned}
&= 0 + \frac{12}{n^2} + \frac{24}{n^2} + \dots + \\
&\frac{12}{n^2} (n-1) \\
&= \frac{12}{n^2} (1 + 2 + \dots + n - \\
&1) \left(\sum_{x=1}^n x = \frac{n(n-1)}{2} \right) \\
&= \frac{12}{n^2} \left(\frac{n(n-1)}{2} \right) \\
&= \frac{6(n-1)}{n} = \frac{6n}{n} - \frac{6}{n} = 6 - \frac{6}{n}
\end{aligned}$$

$$\int_{\underline{6}} f = \sup[R(f, \Pi)] = \sup \left[6 - \frac{6}{n} : n \in N \right] =$$

6 تكامل ريمان الأسفل

الآن نجد المجموع الأعلى ثم تكامل من الأعلى

Now,

$$\begin{aligned}
\bar{R}(f, \Pi) &= \sum_{i=0}^{n-1} M_i |J_i| \\
&= M_0 (x_1 - x_0) + M_1 (x_2 - x_1) + \dots + \\
&M_{n-1} (x_n - x_{n-1}). \\
&= f(x_1) \cdot \frac{2}{n} + f(x_2) \cdot \frac{2}{n} + \dots + f(x_n) \cdot \frac{2}{n} \\
&= \frac{6}{n} \cdot \frac{2}{n} + \frac{12}{n} \cdot \frac{2}{n} + \dots + \frac{6n}{n} \cdot \frac{2}{n} \\
&= \frac{12}{n^2} + \frac{24}{n^2} + \dots + \frac{12n}{n^2} \\
&= \frac{12}{n^2} (1 + 2 + \dots + n) \cdot \left(\sum_{x=1}^n x = \frac{n(n+1)}{2} \right)
\end{aligned}$$

$$= \frac{12}{n^2} \cdot \frac{n(n+1)}{2} = \frac{6(n+1)}{n} = 6 + \frac{6}{n}, \forall n$$

$$\overline{\int} f = \inf[\overline{R}(f, \Pi)] = \inf\left[6 + \frac{6}{n} : n \in \mathbb{N}\right] = 6$$

$$\therefore \overline{\int} f = \underline{\int} f = 6 = \int_0^2 f(x) = 6$$

بما ان تكامل ريمان من الأعلى يساوي تكامل ريمان من الأسفل
فالدالة قابلة للتكامل الريماني

Not:

$$\sum_{x=0}^n x^2 = \frac{n(n+1)(2n+1)}{6}$$

Lecher 9:

Theorem: if f is bounded function defined on $[a, b]$ then f is R.I iff $\forall \epsilon > 0, \overline{R}(f, \Pi) - \underline{R}(f, \Pi) < \epsilon$.

Theorem: Let f is a continuous function on $[a, b]$, then f is Reiman integrable.

Proof:

Let $\epsilon > 0$, since f is continuous on $[a, b]$, then f is uniformly continuous, then
 $\exists \delta > 0$ s.t $\forall x, y \in [a, b]$.

$$|f(x) - f(y)| < \frac{\epsilon}{b-a}$$

By Arch. Property $\exists n \in \mathbb{N}$ s.t $n\delta > b - a \implies \frac{b-a}{n} < \delta$.

$\implies \Pi = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition of $[a, b]$.

$$\text{s.t } |J_i| = \frac{b-a}{n}, J_i = [x_i, x_{i+1}]$$

$$\text{So, } |J_i| = |x_{i+1} - x_i| < \delta, \forall i$$

f is bounded and has max, min. Values in $[a, b]$.
(Since f is continuous on $[a, b]$)

and in $J_i, \forall i$

$$\text{So, } \exists t_i, t_i^* \text{ in } J_i \text{ s.t } M_i = \sup[f(x) : x \in J_i] = f(t_i)$$

$$m_i = \inf[f(x) : x \in J_i] = f(t_i^*)$$

$$\text{Since } |t_i - t_i^*| < \delta, \text{ then } |f(t_i) - f(t_i^*)| < \frac{\epsilon}{b-a}$$

$$\therefore \bar{R}(f, \Pi) - \underline{R}(f, \Pi) = \sum_{i=0}^{n-1} M_i |J_i| - \sum_{i=0}^{n-1} m_i |J_i|$$

$$= \sum (f(t_i) - f(t_i^*)) |J_i|$$

$$= \sum_{i=0}^{n-1} \frac{\epsilon}{b-a} |J_i|$$

$$= \frac{\epsilon}{b-a} \sum_{i=0}^{n-1} |J_i| =$$

$$\frac{\epsilon}{b-a} \cdot (b - a) = \epsilon$$

So f is R.I

* **The Converse of above theorem is not true.**

Example: Let $f(x) = \begin{cases} 3 & \text{if } x \leq 0 \\ 7 & \text{if } x > 0 \end{cases}$, defined on $[-5, 5]$, f is Riemann integrable but not continuous.

Solution:

$$\Pi = \left\{ -5, -\frac{1}{n}, \frac{1}{n}, 5 \right\}, \forall n.$$

$$J_0 = \left[-5, -\frac{1}{n} \right]$$

$$J_1 = \left[-\frac{1}{n}, \frac{1}{n} \right]$$

$$J_2 = \left[\frac{1}{n}, 5 \right]$$

$$\underline{R}(f, \Pi) = \sum_{i=0}^{n-1} m_i |J_i| = m_0 |J_0| + m_1 |J_1| + m_2 |J_2|$$

$$= f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + f(x_2)(x_3 - x_2)$$

$$= 3 \left(-\frac{1}{n} - (-5) \right) + 3 \left(\frac{1}{n} - \left(-\frac{1}{n} \right) \right) +$$

$$7 \left(5 - \frac{1}{n} \right)$$

$$= -\frac{3}{n} + 15 + \frac{3}{n} + \frac{3}{n} + 35 - \frac{7}{n}$$

$$= 50 - \frac{4}{n}$$

$$\underline{\int} f = \sup \left(\underline{R}(f, \Pi) \right)$$

$$= \sup \left[50 - \frac{4}{n} : n \in N \right] = 50$$

$$\overline{R}(f, \Pi) = \sum_{i=0}^{n-1} M_i |J_i|$$

$$= M_1 |J_0| + M_1 |J_2| + M_3 |J_2|$$

$$= M_0(x_1 - x_0) + M_1(x_2 - x_1) + M_2(x_3 - x_2)$$

$$= f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + f(x_3)(x_3 - x_2)$$

$$= f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + f(x_3)(x_3 - x_2)$$

$$= 3\left(-\frac{1}{n} - (-5)\right) + 7\left(\frac{1}{n} - \left(-\frac{1}{n}\right)\right) +$$

$$7\left(5 - \frac{1}{n}\right)$$

$$= \frac{3}{n} + 15 + \frac{7}{n} + \frac{7}{n} + 35 - \frac{7}{n}$$

$$= -\frac{3}{n} + 15 + \frac{7}{n} + 35$$

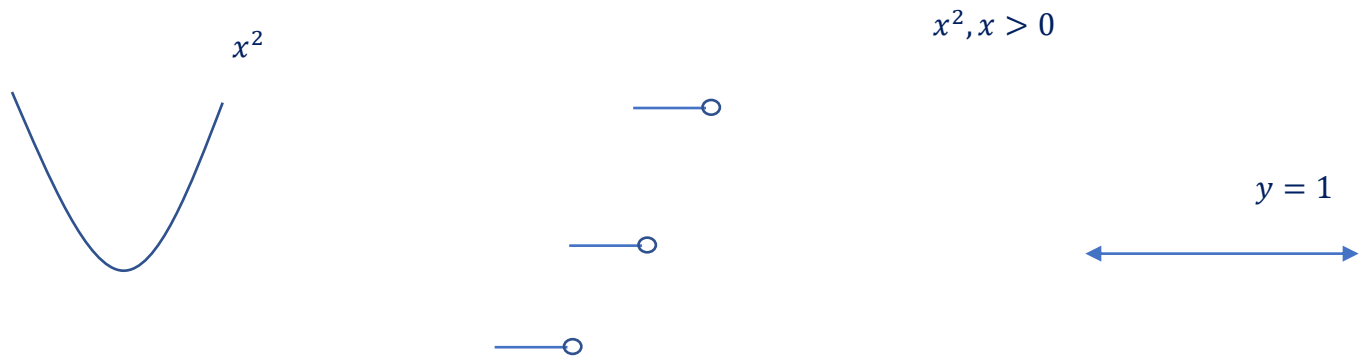
$$= 50 + \frac{4}{n}$$

$$\overline{\int} f = \inf \left(\overline{R}(f, \Pi) \right) = \inf \left[50 + \frac{4}{n} \right] = 50$$

$$\therefore \overline{\int} f = \underline{\int} f \Rightarrow \int_{-5}^5 f(x) = 50$$

Theorem: if f is bounded monotonically (non-decreasing) on $[a, b]$, Then f is R.I





غير رتيبة و مستمرة

رتيبة و غير مستمرة

رتيبة و مستمرة

رتيبة و مستمرة

Proof:

Let $\epsilon > 0$, $\Pi = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition on $[a, b]$, such that

$$J_i = [x_i, x_{i+1}] \text{ and } |J_i| = \frac{b-a}{n}$$

Since f is non-decreasing, then $M_i = f(x_{i+1})$ and $m_i = f(x_i)$

$$\begin{aligned} \overline{R}(f, \Pi) &= \sum_{i=0}^{n-1} [f(x_{i+1}) - f(x_i)] |J_i| \quad \forall i \\ &= [f(b) - f(a)] \left(\frac{b-a}{n} \right) \end{aligned}$$

By Arch. Property, $\exists n \in \mathbb{N}$, such that

$$n\epsilon > (f(b) - f(a))(b - a)$$

$$\Rightarrow (f(b) - f(a)) \frac{(b-a)}{n} < \epsilon$$

$$\text{Since } \overline{R}(f, \Pi) - \underline{R}(f, \Pi) = [f(b) - f(a)] \frac{b-a}{n} < \epsilon$$

$$\therefore \overline{R}(f, \Pi) - \underline{R}(f, \Pi) < \epsilon$$

by theorem (**Let f be a continuous function on $[a, b]$, then f is R.I**)

Lecher 10:

Theorem: if f is bounded and R.I on $[a, b]$, $f(x) \geq 0 \forall x \in [a, b]$. Then $\int f \geq 0$

Definition: Let $S \subset R$, S is called negligible set, if there exist a countable family open intervals $\{I_k\}$ such that

1. $S \subseteq \bigcup_{k=1}^{\infty} I_k$
2. $\forall \epsilon > 0, \sum_{k=1}^{\infty} |I_k| < \epsilon$

نقول ان S مجموعة مهملة اذا امكن تغطيتها بطائفة $\{I_k\}$ من الفترات المفتوحة والتي يقترب مجموع اطوالها من الصفر.

Example: Any finite set of reals is negligible.

Proof:

$$\text{Let } S = \{x_1, x_2, \dots, x_n\}, \epsilon > 0, \forall k, 1 \leq k \leq n$$

Let I_k be an open intervals with center x_k and

$$|I_k| = \frac{\epsilon}{2n}$$

So, $S \subset \bigcup_{k=1}^n I_k$

$$\text{and } \sum_{k=1}^n |I_k| = \sum_{k=1}^n \frac{\epsilon}{2n} = \frac{\epsilon}{2n} n = \frac{\epsilon}{2} < \epsilon$$

$\therefore S$ is negligible set.

* **Any Countable of reals is negligible.**

Proof:

Let $\epsilon > 0$, $\forall k$, I_k open interval with center x_k

$$\text{And } |I_k| = \frac{\epsilon}{2^{k+1}} = \epsilon \sum_{k=1}^{\infty} \frac{1}{2^{k+1}}$$

هذه متسلسلة هندسية وان $r = \frac{1}{2}$ و $a = \frac{1}{2}$ لذا فان

$$\epsilon \sum \frac{1}{2^{k+1}} = \frac{\epsilon}{2} < \epsilon$$

* **Any subset of negligible set is negligible**

* **The union of family of negligible set is negligible**

* **Q is negligible (Since Q is countable)**

* **Any interval $[a, b]$, (a, b) , $[a, b)$ **not** negligible.**

(since its leuguth = $b - a$)

* **R is not negligible, Since its length goes to ∞ .**

* **Q' is not negligible**

* **There is an Uncountable negligible set which is**

Cantor set

Theorem: (Lebesgu's Theorem for R.I)

مبرهنة ليبيغ في التكامل الريماني.

Let f a bounded function on $[a, b]$, then f is Reiman Integrable if and only if the set of discontinuous points of f is negligible set.

f قابلة للتكامل الريماني اذا وفقط اذا كانت مجموعة نقاط عدم الاستمرارية لـ f مجموعة مهملة.

Example: Use Lebesgu's theorem to show f is R.I, where $f: [-4,7] \rightarrow R$,

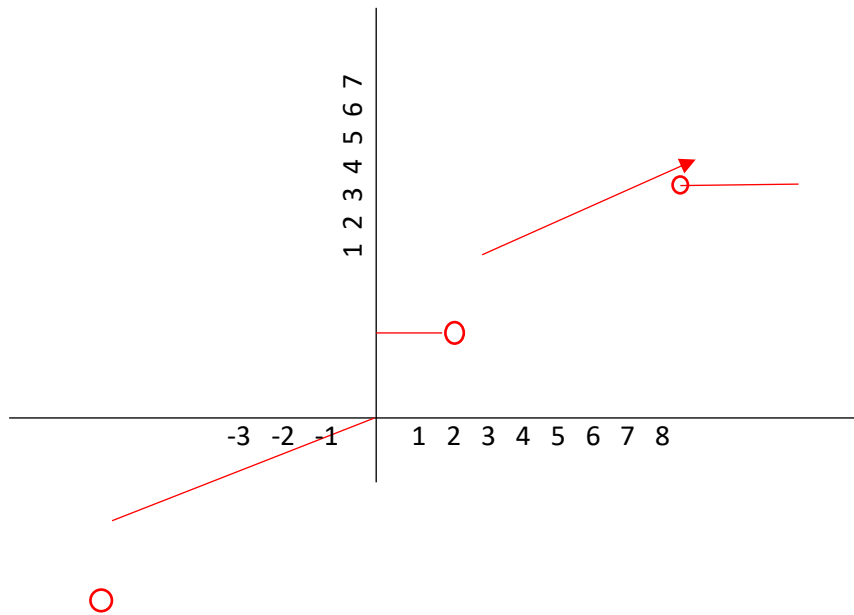
$$f(x) = \begin{cases} x & \text{if } -4 \leq x < 0 \\ 2 & \text{if } 0 \leq x < 2 \\ x + 1 & \text{if } 2 \leq x \leq 5 \\ 5 & \text{if } 5 < x \leq 7 \end{cases}$$

Solution:

f is bounded since

$$|f(x)| \leq 6, \forall x \in [-4,7]$$

and the set of discontinuous points is $\{0, 2, 5\}$ is finite then it is negligible by "Theorem (Lebesgue's theorem for R.I)". f is R.I



Some Properties of Riemann Integration:

1. If f and g are Riemann Integrable on $[a, b]$, $c \in \mathbb{R}$ then $f + g$, $c \cdot f$ are R.I and

$$\text{I. } \int_a^b (f + g) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

$$\text{II. } \int_a^b (c \cdot f)(x) dx = c \int_a^b f(x) dx$$

Proof:

To prove that $f + g, cf$ are R.I
 Since f is R.I, then the set of discontinuity of f is negligible set.
 and g is R.I, then the set of discontinuity of g is negligible set
 (by Leb. Th. for R.I)
 So, the set of discontinuous point of $f + g$ is negligible set, then $f + g$ is R.I

Remark: Let $R.I [a, b]$ = the set of all Riemann integrable function on $[a, b]$, then by (i) and

(ii), \int_a^b is Linear transformation تحويل خطي

$$\int_a^b : R.I [a, b] \rightarrow R, \int_a^b f \in R, \forall f \in R.I [a, b]$$

Lecher 11:

2. If f is Riemann Integrable on $[a, b]$, $f(x) \geq 0, \forall x$ then $\int f \geq 0$

Proof:

Since $f(x) \geq 0, \forall x \in [a, b]$

Then $\underline{R}(f, \Pi) \geq 0$ and $\overline{R}(f, \Pi) \geq 0$, for any partition Π of $[a, b]$

$\Rightarrow \underline{\int} f \geq 0$ and $\overline{\int} f \geq 0$

Since f is R.I, then $\underline{\int} f = \overline{\int} f = \int f \geq 0$

3. If f_1 and f_2 are R.I on $[a, b]$ and, if $f_1 \geq f_2$, then $\int f_1 > \int f_2$

Proof:

Let $g = f_1 - f_2$

$\therefore f_1(x) \geq f_2(x) \rightarrow (f_1(x) = f_1, f_2(x) = f_2)$

$f_1(x) - f_2(x) \geq 0, \forall x \in [a, b]$

$\int g(x) \geq 0$ by property (2)

$$\int_a^b g(x) = \int_a^b f_1 - f_2 \geq 0$$

$$= \int_a^b f_1 + (-f_2) \geq 0$$

$$= \int_a^b f_1 + \int_a^b -f_2 \geq 0 \text{ by (1)}$$

(i and ii)

Then $\int f_1 \geq \int f_2$

4. If f is R.I, then $|f|$ is R.I and

$$|\int f| \leq \int |f|$$

5. If f is R.I then f^2 is R.I

6. If f and g are R.I then the product $f.g$ is R.I

7. $\int: R.I [a, b] \rightarrow R$ is not (one to one) function.

We must prove that for some f, g and $\int f = \int g$ but $f \neq g$?

Example: let $f, g \in RI[-1,1]$, $f: [-1,1] \rightarrow R$, $g: [-1,1] \rightarrow R$ we define f, g as function.

$$f(x) = x, \forall x$$

$$g(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 5 & \text{if } x = 1 \end{cases}$$

Then $\int_{-1}^1 f(x)dx = 0$ and

$$\int_{-1}^1 g(x)dx = 0$$

but $f \neq g$

8. If f continuous on $[a, b]$, $f(x) \geq 0, \forall x$ and $\int f(x) = 0$ then $f = 0$
(means $f(x) = 0, \forall x$)

Proof:

Suppose $f(x) \neq 0$, then $\exists x \in [a, b]$ s.t
 $f(x) \neq 0, y = f(x) > 0$ let $\left(\frac{y}{2}, \frac{3y}{2}\right)$ be
open interval contains y .

Since f is continuous, then $\exists V$ open
interval in \mathbb{R} , such that $x \in V$ and

$$f(V \cap [a, b]) \subseteq \left(\frac{y}{2}, \frac{3y}{2}\right)$$

$$\Rightarrow V \cap [a, b] \subseteq f^{-1}\left(\frac{y}{2}, \frac{3y}{2}\right)$$

$\Rightarrow \exists I$ closed interval s.t $x \in I \subseteq [a, b]$
and $f(x) > 0, \forall x$

And since f is continuous, then f has
min.Value in I say

$$\begin{aligned}
m &\Rightarrow m = \min[f(x) : x \in I] \\
&\Rightarrow m > 0 \\
&\Rightarrow f(x) \geq 0 \\
&\Rightarrow \int_a^b f \geq \int_I f \geq m|I| > a \rightarrow C! \text{ Since} \\
&\int f = 0
\end{aligned}$$

Theorem: if f is R.I on $[a, b]$ and $F(x) = \int_a^x f(t)dt$, then F is Continuous on $[a, b]$

Proof:

Since f is R.I, then f is bounded on $[a, b]$ (by def).

$$\Rightarrow \exists M > 0, \text{ s.t } |f(x)| \leq M, \forall x \in [a, b]$$

$$|F(x) - F(c)| = \left| \int_a^x f(t)dt - \int_a^c f(t)dt \right|$$

$$= \left| \int_c^x f(t)dt \right| \leq \left| \int_c^x |f(t)|dt \right| \leq$$

$$\left| \int_c^x Mdt \right| = |M(x - c)| = M|x - c|$$

Now, for given $\epsilon > 0$, Choose $\delta = \frac{\epsilon}{M}$, then

if $|M(x - c)| < \delta$, then

$$|F(x) - F(c)| \leq M|x - c| < M \cdot \delta < \epsilon$$

$\Rightarrow F$ is continuous at $c, \forall x \in [a, b] \Rightarrow F$ is continuous on $[a, b]$.

Throrem: (Fumdamental Theorem of Calculus)

المبرهنة الأساسية في التفاضل.

If f is R.I on $[a, b]$ and $F(x) = \int_a^x f(t)dt$ and f is Continuous on $[a, b]$, then F is differentiable on (a,b) and $F' = f$

Example: Let f be R.I on $[a, b]$, if $F(x) = \int_a^x f(t)dt$ and F is diff on (a,b) s.t $F' = f$
show that $\int_a^b f(t)dt = F(b) - F(a)$

Theorem: (the integral Mean Value Theorem)

مبرهنة القيمة المتوسطة في التكامل.

Let f and g be continuous on $[a, b]$ with $g(x) \geq 0$ for $x \in [a, b]$, then there is $c \in [a, b]$ such that.

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx$$

Proof:

Since f is Continuous on $[a, b]$, then f is bounded

$\Rightarrow \exists m, M > 0$, s.t $m \leq f(x) \leq M$, $\forall x \in [a, b]$

Since $g(x) \geq 0 \Rightarrow mg(x) \leq f(x)g(x) \leq Mg(x)$, $\forall x$

$$\begin{aligned} \Rightarrow m \int_a^b g(x)dx &\leq \int_a^b f(x)g(x)dx \\ &\leq M \int_a^b g(x)dx \end{aligned}$$

$$\int_a^b g(x)dx = 0$$

If $\int_a^b g(x)dx \neq 0 \Rightarrow \int_a^b f(x)g(x)dx = 0 \Rightarrow$

$$f(c) \int_a^b g(x)dx$$

If $\int_a^b g(x)dx \neq 0 \Rightarrow m \int_a^b g(x)dx \leq$

$$\int_a^b f(x)g(x)dx \leq M \int_a^b g(x)dx$$

$$\Rightarrow m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M$$

Apply (I.V.P) intermediate Value property on f and $[a, b]$ with.

$$k = \frac{\int_a^b fg}{\int_a^b g} \Rightarrow \exists c \in (a, b) \text{ s.t } f(c) = k$$

$$\begin{aligned} \therefore f(c) &= \frac{\int_a^b fgdx}{\int_a^b gdx} \Rightarrow \int_a^b f(x)g(x)dx \\ &= f(c) \int_a^b g(x)dx \end{aligned}$$

Lecher 12:

chapter eight

Measure theory and Lebesgue's Integral

Let $I = (a, b)$ be open interval, We denote the length (Measure) of I by $\Delta(I)$ and $\Delta(I) =$ non-negative real number, then Δ is a mapping on a family of open intervals in \mathbb{R} .

Since every open set in \mathbb{R} is the union of countable open intervals, (if G open in $\mathbb{R} \Rightarrow G = \cup I_k$), then we can define the length of an open set in \mathbb{R} by the **sum** of the length of $I_k, \forall k$.

ما معنى الطول (القياس) لأي مجموعة أخرى؟

وفي الحقيقة العالم الرياضي Lebesgue في عام 1875 – 1941 وجد مجموعة من المجموعات القابلة للقياس في هذه الحالة يمكننا القول

ان توسيع مفهوم الاعداد الحقيقية ومفهوم الحجم للشكل المنتظم الى شكل غير منتظم.

(1) The length of bounded open interval.

let $I = (a, b)$ open interval and $\Delta =$ the set of all open bounded intervals in \mathbb{R} . since $\mathcal{A} = \{I: I \text{ bounded open interval}\}$. we define the following mapping.

$$\Delta: \mathcal{A} \rightarrow \mathbb{R}^+, \text{ s.t } \Delta(I) = \begin{cases} b - a & \text{if } I = (a, b) \\ 0 & \text{if } I = \emptyset \end{cases}$$

$\Delta(I)$ is called the length (measure) of I .

" properties of " Δ

1. For any I , $\Delta(I) \geq 0$
2. If $I_1 \subseteq I_2$, then $\Delta(I_1) \leq \Delta(I_2)$, $I_1, I_2 \in \mathcal{A}$
3. $\Delta(I_1 \cup I_2) \leq \Delta(I_1) + \Delta(I_2)$
4. If $I_1 \cap I_2 = \emptyset$, then $\Delta(I_1 \cup I_2) = \Delta(I_1) + \Delta(I_2)$
5. If $I \subseteq \bigcup_{k=1}^n I_k$, then $\Delta(I) \leq \sum_{k=1}^n \Delta(I_k)$
6. In general, if $I \subseteq \bigcup_{k=1}^{\infty} I_k$, then $\Delta(I) \leq \sum_{k=1}^{\infty} \Delta(I_k)$
7. let $\delta \in \mathbb{R}, I \in \mathcal{A}$. If we define the set $\delta + I = [\delta + x: x \in I]$,

(translation of I , $|s|$ units) then

$$\Delta(S + I) = \Delta(I)$$

Proof:(1)

by definition, $0 \leq \Delta(I) = \begin{cases} b - a \\ 0 \end{cases}$

proof:(2)

let $I_1 = (a, b)$, $I_2 = (c, d)$, and $I_1 \subseteq I_2$.

$$\begin{aligned} \Delta(I_1) &= (d - c) - (a - c) - (d - b) \\ &= \Delta(I_2) - [(a - c) + (d - b)]. \end{aligned}$$

since $(a - c) + (d - c) \geq 0$, then

$$\Delta(I_1) \leq \Delta(I_2)$$

proof:(3)

$I_1 = (a, b)$, $I_2 = (c, d)$.

$$\begin{aligned} \Delta(I_1 \cup I_2) &= b - a + (d - c) - (b - c) \\ &= \Delta(I_1) + \Delta(I_2) - \Delta(I_1 \cap I_2) \geq 0 \\ &\leq \Delta(I_1) + \Delta(I_2). \end{aligned}$$

proof:(4)

If $I_1 \cap I_2 = \emptyset$, then $\Delta(I_1 \cup I_2) = \Delta(I_1) + \Delta(I_2) - \Delta(I_1 \cap I_2) = \Delta(I_1) + \Delta(I_2)$

Proof:(5)

الاستقراء

The prove follows from induction and (2),(3)

Proof:(6)

let $I = (a, b)$, I_k open interval, $k \leq 1, 2, \dots$

If $I \neq \phi \rightarrow \Delta(I) = \sum_{k=1}^{\infty} \Delta(I_k)$

Since $I \subseteq \bigcup I_k$.

$\Rightarrow \{I_k : k = 1, 2, \dots\}$ open Cover for I .

let $\epsilon > 0$ and $I^* = (a - \epsilon, a + \epsilon)$ Cover for a .

$I^{**} = (b - \epsilon, b + \epsilon)$ Cover for b .

$\Rightarrow \bigcup_{k=1}^{\infty} I_k \cup I^* \cup I^{**}$ open Cover for $[a, b]$.

$\Rightarrow \bigcup_{k=1}^{\infty} I_k \cup I^* \cup I^{**}$ open Cover for $[a, b]$.

since $[a, b]$ is Compact, then this Cover has a finite

snbcover $\Rightarrow \exists I_1, I_2, \dots, I_n$ s.t

$I^* \cup I^{**} \cup \bigcup_{k=1}^n I_k$ cover for $[a, b]$

$\Rightarrow I \subset \bigcup_{k=0}^n I_k \cup I^* \cup I^{**}$

$\Delta(I) \leq \Delta(I^*) + \Delta(I^{**}) + \sum_{k=1}^n \Delta(I_k)$

$= 2\epsilon + 2\epsilon + \sum_{k=1}^n \Delta(I_k)$

$= 4\epsilon + \sum_{k=1}^n \Delta(I_k)$

$\leq 4\epsilon + \sum_{k=1}^{\infty} \Delta(I_k)$

this is true for all $\epsilon > 0$, then

$$\Delta(I) \leq \sum_{k=1}^{\infty} \Delta(I_k)$$

Proof:(7)

let $I = (a, b)$, then $s + I = (s + a, s + b)$

$$\Delta(s + I) = s + b - (s + a) = b - a = \Delta(I)$$

(2) The length of Bounded open sets

اطوال المجموعات المفتوحة المقيدة

lemma: Let $G \subseteq R$, if G is open set, then $G =$ union of Countable family of disjoint bounded open intervals

اتحاد عناصر عائلة معدودة من الفترات المفتوحة المقيدة المنفصلة

Also, the representation of G by this way will be unique

i.e: $\exists! \{I_i: I_i \text{ open bounded interval}, i = 1, 2, \dots\}$.

s. t $G = \bigcup_{i=1}^{\infty} I_i$ and $I_i \cap I_j = \emptyset, \forall i, j$

Definition: Let G be a bounded open set, then the length of G , $\Delta(G) = \sum_{i=1}^{\infty} \Delta I_i$, where $\{I_i\}$ is a unique family of open bounded intervals, s.t $G = \bigcup I_i$ and $I_i \cap I_j = \emptyset$.

Question: Does \sum the Series $\sum_{i=1}^{\infty} \Delta I_i$ in the above definition is Convergent?

Yes, let $S_m = \sum_{n=1}^m \Delta I_n$, $\langle S_m \rangle$ the sequence of partial sums of $\sum \Delta I_n$, since G is bounded, then $\exists I$

open interval s.t $G \subset I$

$\Rightarrow \bigcup_{n=1}^{\infty} I_n \subset I \Rightarrow \sum_{n=1}^{\infty} \Delta I_n \leq \Delta(I) \Rightarrow S_m =$

$\sum_{n=1}^m \Delta(I_n) \leq \sum_{n=1}^{\infty} \Delta I_n$

$\Rightarrow \langle S_m \rangle$ is bounded and since $\Delta(I_n) \geq 0$,

then $\langle S_m \rangle$ is non-decreasing $\Rightarrow \langle S_m \rangle$ Convergent.

i.e: $\sum_{n=1}^{\infty} \Delta I_n$ Convergent.

Properties of Δ :

Let β = the set of all bounded open sets
= $\{G: G \text{ bounded open set}\}$

$$\Delta: \beta \rightarrow R^+ \text{ s.t } \Delta(G) = \begin{cases} \sum_{i=1}^{\infty} \Delta(I_i) & \text{if } G \neq \phi \\ 0 & \text{if } G = \phi \end{cases}$$

Then:

1. $\Delta(G) \geq 0$
2. If $G_1 \subset G_2$, then $\Delta(G_1) \leq \Delta(G_2)$
3. $\Delta(G_1 \cup G_2) = \Delta(G_1) + \Delta(G_2) - \Delta(G_1 \cap G_2)$
and, if $G_1 \cap G_2 = \phi$, then $\Delta(G_1) + \Delta(G_2)$
4. If $\{G_n\}$ countable family, then
 $\Delta(\cup G_n) \leq \sum \Delta(G_n)$
and if $\{G_n\}$ disjoint, then $\Delta(\cup G_n) = \sum \Delta(G_n)$
5. $\Delta(S + G) = \Delta(G)$, $S \in R$, where
 $S + G = \{s + x : x \in G\}$

Remark:

1. If G bounded , then $S + G$ is bounded
2. If G open, then $S + G$ is open

Since the properties of a set does not chang by translation.

Proof:

1. G is bounded, then

$$\exists M > 0 \quad s.t \quad d(x, y) = |x - y| \leq M.$$

$$\Rightarrow d(s + x, s + y) = |s + x - (s + y)| = |x - y| \leq M, \forall x, y$$

$\therefore S + G$ bounded

2. If G is open, then $G =$

$$\bigcup_{k=1}^{\infty} I_k, I_k \text{ is open}$$

interval and $I_k = (a_k, b_k)$

$$S + G = \bigcup (s + I_n)$$

$$= \bigcup (S + a_k, s + b_k)$$

(3) Outer Measur of bounded sets

القياس الخارجي للمجموعة المقيدة

Defition: Let S bounded subset of \mathbb{R} and

$\theta(s) = \{G : G \text{ open bounded set and } S \subset G\}$

than $M^*(s) = \inf\{\Delta(G) : G \in \theta(s)\}$ is called the outer Measure of S .

القياس الخارجي للمجموعة S هو اصغر قياس لاصغر مجموعة تحوي S او اكبر قيد لمجموعة قياسات المجموعة المفتوحة التي تحوي S

Remark:

1. $M^*(s)$ = the Measure of the smallest open set G containing s .
2. $\forall (s) = \emptyset$, since s is bounded $\implies \exists (a, b)$ open interval and $S \subseteq (a, b)$
3. The set $\{\Delta(G) : G \in \theta(s)\}$ is bounded below by Zero so **inf exists**

Example:

1. If S is bounded open set, then $M^*(s) = \Delta(s)$, since $M^*(s) = \inf\{\Delta(G) : G \in \theta(s)\}$ since $s \in \theta(s)$ and $s \subseteq S$, then $\inf\{\Delta(G) : G \in \theta(s)\} = \Delta(s)$
2. $M^*(\emptyset) = \Delta(\emptyset) = 0$
3. If $S = \{x\}$, then $M^*(s) = 0$

Proof:

$\forall n \in \mathbb{N}, G_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ is open

bounded interval and $S \subset \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \forall n$
 $\Delta(G_n) = \frac{2}{n} \rightarrow 0$ as $n \rightarrow \infty$

then $M^*(s) = \inf\{\Delta(G_n); n \in N\} =$
 $\inf\left\{\frac{2}{n}; n \in N\right\} = 0$

4. If S is countable set $M^*(s) = 0$.

Proof:

Let $S = \{x_1, x_2, x_3, \dots\}$ and $G_n =$
 $\left(x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}}\right), \epsilon > 0 \rightarrow x_n \in G_n$
 $\rightarrow S \subset \cup G_n$

$$\Delta(G_n) = \frac{2\epsilon}{2^{n+1}} = \frac{\epsilon}{2^n}$$

$$\Delta(G_n) \leq \sum_{n=1}^{\infty} \Delta(G_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$$

(since $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$) Geometric series.

$$\begin{aligned} M^*(s) &= \inf\{\Delta(\cup G_n): n = 1, 2, \dots\} \\ &= \inf\{\sum_{n=1}^{\infty} \Delta(G_n) : G \in \theta(s)\} \\ &= \inf\{\epsilon : \epsilon > 0\} = 0 \end{aligned}$$

5. If S is negligible, then $M^*(s) = 0$

6. If S is bounded set and $M^*(s) = 0$, then S is negligible set

Proof:

Let $\epsilon > 0, M^*(S) = 0 \rightarrow \inf\{\Delta(G): S \subset G\} =$
 0

$\rightarrow \forall \epsilon > 0, \exists G$ s.t $S \subset G$ and $\Delta(G) < \epsilon$, G is bounded open by Lemma $G = \cup I_n$,

$I_i \cap I_l = \emptyset$, I open bounded interval

$\rightarrow \Delta(G) = \sum_{n=1}^{\infty} \Delta(I_n) < \epsilon$

$\therefore S \subseteq I_n$, and $\sum \Delta(I_n) < \epsilon \rightarrow S$ is negligible

7. If

$S = [a, b]$ or (a, b) or $(a, b]$, then $M^*(s) = b - a$

Proof:

Let $G_n = \left(a - \frac{1}{n}, b + \frac{1}{n}\right) \in N$

$\Delta(G) = b - a - \frac{2}{n}, n \in N$

$S \subset \cup G_n$

$\therefore M^* = \inf\{\Delta(G_n) : n \in N\}$

$= \inf\left\{b - a + \frac{1}{n} : n \in N\right\}$

$= b - a$

Properties of M^*

Let $C = \{S \subseteq R, S \text{ is bounded}\}$

$M^*: C \rightarrow R^+$ s.t $M^*(s)$

$= \begin{cases} \inf\{\Delta(G) : S \subset G & \text{if } S \neq \emptyset \\ 0 & \text{if } S = \emptyset \end{cases}$

Then:

1. $M^*(s) \geq 0, S \subseteq C$

2. If $S_1 \subseteq S_2$, then $M^*(s_1) \leq M^*(s_2)$; $s_1, s_2 \in \mathcal{C}$
3. $M^*(S_1 \cup S_2) \leq M^*(S_1) + M^*(S_2)$
4. If $\{S_n\}$ countable family s.t $S_n \in \mathcal{C}$, $\forall n$, then $M^*(\cup S_n) \leq \sum M^*(S_n)$
5. If $t \in \mathbb{R}$, $S \in \mathcal{C}$ then $M^*(t + s) = M^*(s)$

Lecher 14:

(4) The measure of bounded setes:

Definition: Let E is bounded subset of \mathbb{R} , E is called measureable (قابلة للقياس) set iff

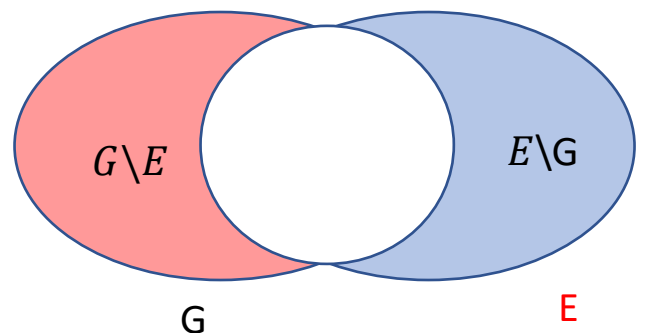
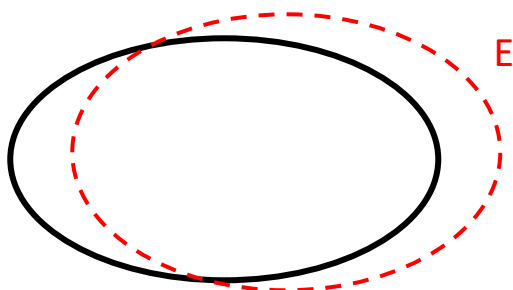
$\forall \epsilon > 0$, $\exists G, G$ is bounded open set, s.t

$$M^* |G - E| < \epsilon$$

Where $|G - E| = \{x \in G \text{ and } x \notin E\} \cup \{x: x \in E \text{ and } x \notin G\}$

$$\text{Or } |G - E| = G \setminus E \cup E \setminus G$$

المجموعة E قابلة للقياس اذا امكن تقريباها لمجموعة مقيدة مفتوحة و عليه يكون الرسم اكثر دقة كالاتي:



If E measurable set then $M(E) = M^*(E)$ and $M(E)$ is called the measure of E .

Theorem: Let $E \subset \mathbb{R}^n$ is bounded, E is called measurable iff $\forall \epsilon > 0, \exists G$ bounded open set s.t $E \subset G$

Example:

1. If E bounded open then E is Measurable and $M(E) = \Delta(E)$

Proof:

$$\begin{aligned} \text{take } E = G &\rightarrow |G - E| = \emptyset \\ &= M^* = |G - E| = 0 < \epsilon, \forall \epsilon \\ \therefore E &\text{ measurable.} \end{aligned}$$

2. If E is negligible, then measurable and $M(E) = 0$

Proof:

E is negligible than $\exists \{I_n\}$ countable family of open interval set $E \subset \cup \{I_n\}$ and $\sum \Delta(I_n) < \epsilon$
 $\rightarrow \cup \{I_n\}$ open and bounded
 take $G_n = \cup I_n$

$\therefore M^*(\cup I_n - E) < \epsilon \rightarrow E$ is measurable
and $M^*(E) = M(E)$

3. If

$E = [a, b]$ or (a, b) or $[a, b]$, then $M(E) = b - a$

Proof:

If $E = [a, b]$

Take $G = (a, b) \rightarrow G$ is open and bounded.

$\therefore M^*(G - E) = M^*[a, b] = 0 < \epsilon, \forall \epsilon$

$\therefore E$ is Measurable and $M^*(E) = M(E) = b - a$

Propertise of M:

Let $D = \{E: E \text{ is bounded subset of } R\}$

$M: D \rightarrow R^+$ s.t $M(E) = M^*(E)$ if E is measuerable

Then:

- $M(E) \geq 0$

- If $E_1 \subseteq E_2 \implies M(E_1) \leq M(E_2) ; E_1, E_2 \in D$

- $M(E_1 \cup E_2) = M(E_1) + M(E_2)$ if

$$E_1 \cap E_2 = \emptyset \rightarrow M(E_1 \cup E_2) = M(E_1) + M(E_2)$$

- $\{E_n\}$ countable family of bounded sets, the

$$M(\cup E_n) \leq \sum_n M(E_n)$$

في تكامل ريمان نأخذ f دالة مقيدة كشرط أساسي لدالة التكامل. فان سنحاول نلخص هذا الشرط

Definition: let $S \subseteq R$, $f: S \rightarrow R$ be a function, f is called Measurable, if $\forall G$ open set and $f^{-1}(G) \subseteq S$, $f^{-1}(G)$ must be measurable set.

Remark: when $f^{-1}(G) = S$, then f is measurable, if its domain is measurable set.

Example: Dose measurable function?

$$1. f: [a, b] \rightarrow s. t f(x) = \begin{cases} 2 & , \text{if } x \in Q \\ 3 & , \text{if } x \in Q' \end{cases}$$

Let G open R , to find $f^{-1}(G)$?

$$f^{-1}(G) = \begin{cases} [a, b] & , \text{if } 2, 3 \in G \\ \text{rational in } [a, b] & , \text{if } 2 \in G \text{ and } 3 \notin G \\ \text{irrational in } [a, b] & , \text{if } 3 \in G \text{ and } 2 \notin G \\ \emptyset & , \text{if } 2, 3 \notin G \end{cases}$$

توضيح

$$\begin{aligned} M^*((a-b) - f^{-1}) \\ = M^*(\text{rational in } [a, b]) = 0 \\ \therefore M^*(f^{-1}(G)) = M^* = b - a \end{aligned}$$

$$\therefore M^*(f^{-1}(G)) = \begin{cases} b - a & \text{if } 2, 3 \in G \\ 0 & \text{if } 2 \in G \text{ (G is countable)} \\ M^*((a, b)) = b - a & \text{if } 3 \in G \\ 0 & , \text{if } 2, 3 \notin G \end{cases}$$

$\therefore f$ measurable

2. If $f: [a, b] \rightarrow R$ monotone then $f^{-1}(R) = [c, d]$ measurable $\Rightarrow f$ measurable

Lebesgue's Measure for unbounded sets

Definition: The set E is Lebesgue's Measurable, if for each A , we have

$$M^*(A) = M^*(A \cap E) + M^*(A \cap E^c)$$

Remark:

1. Since $A \subseteq (A \cap E) \cup (A \cap E^c) \Rightarrow M^*(A) \leq M^*(A \cap E) + M^*(A \cap E^c)$
2. If E is Measurable, then $M^*(E)$ is denoted as $M(E)$

Measurable functions:

دوال القياس

Definition: Let f be a function defined on $[a, b]$, we call f be a measurable, if $\forall \alpha \in R$, then set $\{x: f(x) > \alpha\}$ is measurable

i.e.: f is measurable $\Leftrightarrow f^{-1}(\alpha, \infty)$ measurable set $\forall \alpha \in R$

Theorem: The function on $[a, b]$ is measurable iff one of the following conditions is hold

1. $\{x : f(x) \geq a\}$ is measurable, $\forall a \in R$
2. $\{x : f(x) > a\}$ is measurable, $\forall a \in R$
3. $\{x : f(x) \leq a\}$ is measurable, $\forall a \in R$
4. $\{x : f(x) < a\}$ is measurable, $\forall a \in R$

Theorem: If f is measurable, then $|f|$ is measurable

Theorem: If f is measurable on $[a, b]$, $r \in R$ then $(r + f)$, (r, f) , $(-f)$ are measurable

Example: Let $f(x) = \begin{cases} x + 5 & , \text{if } x < -1 \\ 2 & \text{if } -1 \leq x \leq 0 \\ x^2 & , \text{if } x > 0 \end{cases}$

show that f is measurable function.

Let $\alpha \in R$, then the set $\{x : f(x) \leq \alpha\}$

$$\begin{aligned}
&= \{x : f(x) \leq \alpha\} \\
&= \begin{cases} (-\infty, \alpha - 5) & , \text{if } \alpha < 0 \\ (-\infty, -5) \cup \{0\} & , \text{if } \alpha = 0 \\ (-\infty, \alpha - 5] \cup [0, \sqrt{\alpha}] & \text{if } 0 < \alpha < 2 \\ (-\infty, \alpha - 5] \cup [-1, \sqrt{\alpha}] & , \text{if } 2 \leq \alpha < 4 \\ (-\infty, \sqrt{\alpha}] & , \text{if } \alpha \geq 4 \end{cases}
\end{aligned}$$

Example: If $f(x) = c$, $\forall x \in R$, then f is measurable since, if $\alpha \in R$, $c \geq \alpha$, then the set $\{x: f(x) \geq \alpha\} = R \rightarrow$ Measurable.

If $a \geq c$, then $\{x: f(x) \geq \alpha\} = \emptyset \rightarrow$ measurable.

Theorem: if f_1 and f_2 are measurable, then $f_1 \pm f_2$, $f_1 \cdot f_2$, f_1/f_2 $f_2 \neq 0$ are measurable

Example: $g(x) = |x| + 2x$ is measurable

Example: Every continuous function is measurable

Let $x \in R$, the set $\{x: f(x) > \alpha\} = f^{-1}(\alpha, \infty)$ by continuity $f^{-1}(\alpha, \infty)$ also open interval then f is measurable.

Lebesgue Integral:

1. f bounded on $[a, b]$

2. f bounded on measurable set of function
measure
3. f unbounded
4. The general definition.

Theorem: Every bounded Riemann integral function on $[a, b]$ is Lebesgue integral and

$$R \int f = L \int f$$

Proof:

If f is Riemann integral, then

$$R \int_{-a}^a f dx \leq R \int_a^b f dx = R \int_a^b f dx$$

by Remark:

$$R \int_a^b f dx \leq L \int_{-a}^b f dx \leq L \int_a^b f dx \leq R \int_a^b f dx$$

Then R.I \Rightarrow L.T but the Converse not true by the following example.

Example: Let $f: [0,1] \rightarrow R, f(x) = \begin{cases} 0 & \text{if } x \in Q \\ 1 & \text{if } x \in Q' \end{cases}$

P partition of $[0,1], P = \{A_1, A_2\}$ where

$$A_1 = \{x \text{ in } [0,1]: x \in Q\} \Rightarrow M(A_1) = 0$$

$$A_2 = \{x \text{ in } [0,1] : x \in Q'\} \Rightarrow M(A_2) = 1$$

(since $A_2 = [0,1] \setminus A_1$)

$$L(f, p) =$$

$$\inf\{f(x)\}_{x \in A_1} \cdot M(A_1) + \inf\{f(x)\}_{x \in A_2} \cdot M(A_2)$$

$$= 0 \cdot 0 + 1 \cdot 1 = 1$$

$$U(f, p) =$$

$$\sup\{f(x)\}_{x \in A_1} \cdot M(A_1) + \sup\{f(x)\}_{x \in A_2} \cdot M(A_2)$$

$$= 0 \cdot 0 + 1 \cdot 1 = 1$$

f is bounded function on $[a, b]$ and

$P = \{A_1, A_2, \dots, A_n\}$ be any partition of $[a, b]$ where A_1, A_2, \dots, A_n are measurable subsets of $[a, b]$ such that

$$\bigcup_{n=1}^n A_i = [a, b]$$

And $M(A_i \cap A_j) = 0$, for $i \neq j$

define

$$U(f, p) = \sum_{i=1}^n \sup_{x \in A_i} \{f(x)\} \cdot M(A_i) \text{ L.}$$

upper sum

$$L(f, p) = \sum_{i=1}^n \inf_{x \in A_i} \{f(x)\} \cdot M(A_i) \text{ L. lower sum}$$

Then $U(f, p) \geq L(f, p)$

$$L \int_a^{-b} f = \inf [U(f, p) : \forall p \text{ partion }]$$

$$L \int_{-a}^b f = \sup [L(f, p) : \forall p \text{ partion }]$$

f is called Lebesgue integrable, if

$$L \int_{-} f = L \int^{-} f = L \int_a^b f$$

NOT: $L \int_{-a}^b \leq L \int_a^{-b}$

Remark: $R \int_{a-}^b \leq L \int_{-a}^b \leq L \int_a^{-b} \leq \int_a^{-b}$

now, $\int_{-a}^b f = \sup [L(f, p)] = \inf [U(f, p)] = \int_a^{-b} f$

$\therefore f$ is Lebesgue integrable

Theorem: Let $f: [a, b] \rightarrow R$, bounded function, f is Lebesgue integrable iff $\forall \epsilon > 0, \exists$ a Lebesgue partition P of $[a, b]$ such that $U(f, p) - L(f, p) < \epsilon$

Theorem: Every bounded measurable function f on $[a, b]$ is Lebesgue integrable on $[a, b]$

